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Splitting Differential Algebraic Groups

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INTRODUCTION

Throughout this paper we use standard terminology of differential algebra from Kolchin's books [K1, K2]. So we denote by $\mathcal{U}$ a universal Δ-field of characteristic zero with field of constants $\mathcal{K}$ and consider a Δ-subfield $\mathcal{F}$ of $\mathcal{U}$ (over which $\mathcal{U}$ is universal) with field of constants $\mathcal{C}$. By a linear Δ-$\mathcal{F}$-group [C1] we mean a Δ-$\mathcal{F}$-closed subgroup G of some $GL_n(\mathcal{U})$. Let us make the following:

DEFINITION. A linear Δ-$\mathcal{F}$-group $G \subset GL_n(\mathcal{U})$ is called split if it is of the form $G = G^* \cap GL_n(\mathcal{K})$, where G^* is a $\mathcal{C}$-closed subgroup of $GL_n(\mathcal{U})$ (G^* coincides then with the $\mathcal{C}$-closure of G in $GL_n(\mathcal{U})$). G is called splittable over an extension $\mathcal{F}_1$ of $\mathcal{F}$ if it is Δ-$\mathcal{F}_1$-isomorphic to a split linear Δ-$\mathcal{F}$-group; it will be called splittable if it is splittable over some extension of $\mathcal{F}$.

Splittable groups naturally appeared in Cassidy's work [C1, C2, C3] on semisimple and unipotent Δ-algebraic groups. To simplify our exposition assume throughout the paper that $\mathcal{F}$ is algebraically closed. Moreover we will concentrate ourselves on irreducible linear Δ-$\mathcal{F}$-groups. Clearly, if such a group G is splittable then tr. deg. $\mathcal{F}\langle G\rangle/\mathcal{F} < \infty$. The converse fails as shown by the example of the Δ-subgroup of $GL_1(\mathcal{U})$ defined by the equation $y''y - (y')^2 = 0$ (cf. (2.2) below). Our aim in this paper is to exhibit a large class of G's for which the converse holds.

Recall that for a linear Δ-$\mathcal{F}$-group $G \subset GL_n(\mathcal{U})$ the set of all Δ-closed normal irreducible solvable subgroups of G has a unique maximal element which obviously is Δ-$\mathcal{F}$-closed and will be called the radical of G. Moreover a linear Δ-$\mathcal{F}$-group $G \subset GL_n(\mathcal{U})$ is called unipotent (cf. [C1]) if it consists of unipotent matrices. Now we can state our

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MAIN THEOREM. *Let G be an irreducible linear Δ-$\mathscr{F}$-group with* tr. deg. $\mathscr{F}\langle G\rangle/\mathscr{F} < \infty$. *If the radical of G is unipotent then G is splittable over some Picard–Vessiot extension of $\mathscr{F}$.*

The "extreme" case when G is semisimple is a special case of a result due to P. Cassidy [C2]; however, we will not use her results here and develop instead a quite different method based on the interplay between differential algebra and the Hopf algebra machinery in [H]. Our method has an interest in itself since it relates splittability with representation theory of Lie algebras and with representative functions. Consequently we will borrow our terminology of affine algebraic groups from [H] (rather than from [K1]).

So $\mathscr{L}(A)$, $\mathscr{L}(\mathscr{G})$ denote the Lie algebra associated to an associative algebra A and to an affine algebraic group $\mathscr{G}$, respectively. $\mathscr{P}(\mathscr{G})$ will denote the affine Hopf algebra associated to $\mathscr{G}$. Moreover $\mathscr{G}(H)$ will denote the affine algebraic group associated to an affine Hopf algebra H; the letter $\mathscr{G}$ will never be used to denote a Δ-field (like in [C1, K1]). To avoid confusion with our universal Δ-field $\mathscr{U}$ we denote by $U(L)$ (rather than $\mathscr{U}(L)$ as in [H]) the universal enveloping algebra of the Lie algebra L. Finally, note that by a Δ-Lie $\mathscr{F}$-algebra we understand a Δ-algebra over $\mathscr{F}$ which is a Lie algebra; this is the concept in [B1] and is different from that of Δ-$\mathscr{F}$-Lie algebra in [C2, K2].

1. FINITE GENERATION

The first step in our approach is the following

(1.1) THEOREM. *Let G be an irreducible linear Δ-$\mathscr{F}$-group with* tr. deg. $\mathscr{F}\langle G\rangle/\mathscr{F} < \infty$. *Then the Δ-coordinate algebra $\mathscr{F}\{G\}$ is finitely generated as a (non-differential) $\mathscr{F}$-algebra.*

The above theorem allows us to consider the affine algebraic $\mathscr{F}$-group $\mathscr{G}(\mathscr{F}\{G\})$, where $\mathscr{F}\{G\}$ is viewed with the natural Hopf Δ-algebra structure induced from that of $\mathscr{F}\{GL_n(\mathscr{U})\}$ via the given embedding $G \subset GL_n(\mathscr{U})$; similarly we get an affine algebraic $\mathscr{U}$-group $\mathscr{G}(\mathscr{U}\{G\})$. Note that G can be naturally seen as a Zariski dense subgroup of $\mathscr{G}(\mathscr{U}\{G\})$ via the identifications

$$G = \mathrm{Hom}_{\Delta\text{-alg}}(\mathscr{U}\{G\}, \mathscr{U})$$

$$\mathscr{G}(\mathscr{U}\{G\}) = \mathrm{Hom}_{\mathrm{alg}}(\mathscr{U}\{G\}, \mathscr{U}).$$

For the proof of (1.1) we need the following lemma (in which $Q(R)$ means the quotient field of the integral domain R):

(1.2) LEMMA. *Let $\mathscr{A} \subset \mathscr{B}$ be a Δ-finitely generated extension of Δ-$\mathbb{Q}$-algebras. Assume $\mathscr{B}$ is an integral domain and* tr. deg $Q(\mathscr{B})/Q(\mathscr{A}) < \infty$. *Then there exists a non-zero element $s \in \mathscr{B}$ such that $\mathscr{B}[1/s]$ is finitely generated over $\mathscr{A}$ as a (non-differential) algebra.*

Proof. Proceeding by induction on the number of Δ-generators of $\mathscr{B}$ over $\mathscr{A}$ we may assume that $\mathscr{B} = \mathscr{A}\{b\}$. Let Θ denote as usual the free commutative monoid built on Δ. If $\theta = \delta_1^{\alpha_1} \cdots \delta_m^{\alpha_m}$ and $\eta = \delta_1^{\beta_1} \cdots \delta_m^{\beta_m}$ we write $\theta < \eta$ if and only if $(\sum \alpha_i, \alpha_1, ..., \alpha_m) < (\sum \beta_i, \beta_1, ..., \beta_m)$ in the lexicographic order. We write $\theta \leqslant \eta$ if either $\theta < \eta$ or $\theta = \eta$. Finally, we write $\theta \subseteq \eta$ if $\alpha_i \leqslant \beta$ for all i. For any $\theta \in \Theta$ put $\mathscr{B}^\theta = \mathscr{A}[\eta b; \eta < \theta]$ and construct inductively (with respect to $\leqslant$) subsets Σ^θ of Θ in the following way: $\Sigma^1 = \varnothing$ if b is algebraic over $\mathscr{B}^1$ and $\Sigma^1 = \{1\}$ if b is transcendental over $\mathscr{B}^1$; if η is the successor of θ put

$$\Sigma^\eta = \begin{cases} \Sigma^\theta & \text{if } \eta b \text{ is algebraic over } \mathscr{B}^\theta \\ \Sigma^\theta \cup \{\eta\} & \text{if } \eta b \text{ is transcendental over } \mathscr{B}^\theta. \end{cases}$$

Put $\Sigma = \bigcup_\theta \Sigma^\theta$, $\Lambda = \Theta \backslash \Sigma$ and let $\Lambda_{\min}$ be the set of minimal elements of Λ with respect to the order "$\subseteq$." Clearly $\Lambda_{\min}$ is a finite set $\{\theta_1, ..., \theta_M\}$. Define $R = \mathscr{A}[\theta b, \theta \in \Sigma]$; it is a polynomial algebra over $\mathscr{A}$ (in finitely many variables), which in general is not a Δ-subalgebra of $\mathscr{B}$. Now for any $i \in \{1, ..., M\}$ let F_i be a non-zero polynomial of minimum degree in $\mathscr{B}^{\theta_i}[T]$ such that $F_i(\theta_i b) = 0$. Since $\mathbb{Q} \subset \mathscr{A}$, $dF_i/dT \neq 0$ and so $s_i = (dF_i/dT)(\theta_i b) \in \mathscr{B}^{\theta_i}[\theta_i b]$, $s_i \neq 0$, hence $s = s_1 \cdots s_M \in \mathscr{B}$ is a non-zero element. Now it is easy to check that $\theta b \in \mathscr{B}^\theta[\theta_i b, 1/s_i]$ for all $1 \leqslant i \leqslant M$ and $\theta_i \subseteq \theta$.

This immediately implies that $\mathscr{B}[1/s] = R[\theta_1 b, ..., \theta_M b, 1/s_1, ..., 1/s_M]$ and we are done.

(1.3) *Proof of Theorem* (1.1). We may assume $\mathscr{F}$ is uncountable. By the above lemma, the scheme $X = \operatorname{Spec} \mathscr{R}(\mathscr{R} = \mathscr{F}\{G\})$ contains an open affine set X_0 of finite type over $\mathscr{F}$. Now X is a group scheme over $\mathscr{F}$. Let $M_1 \in X \backslash X_0$ and look for an affine neighborhood X_1 of M_1 of finite type over $\mathscr{F}$. We may assume M_1 is a maximal ideal. Since $\mathscr{F}$ is algebraically closed, uncountable and $\mathscr{R}/M_1$ is a countably generated $\mathscr{F}$-vector space, a well-known argument shows that $\mathscr{R}/M_1 \simeq \mathscr{F}$, hence $M_1 = \ker g_1$ for some $\mathscr{F}$-point of X, $g_1 \in X(\mathscr{F})$. Now take any $g_0 \in X(\mathscr{F})$ such that $M_0 = \ker g_0 \in X_0$ and conclude by letting X_1 be the image of X_0 via translation from the right with $g_1 g_0^{-1} \in X(\mathscr{F})$.

(1.4.) Note that if we are given a Δ-$\mathscr{F}$-isomorphism $G \to G'$ between linear Δ-$\mathscr{F}$-groups with tr. deg. $\mathscr{F}\langle G\rangle/\mathscr{F} < \infty$ we get an induced birational map from $\mathscr{G}(\mathscr{F}\{G\})$ to $\mathscr{G}(\mathscr{F}\{G'\})$ which agrees with multiplication maps whenever operations make sense. Such a map must be an isomorphism (cf. [L, p. 5]).

2. Splitting

In this section we prove our Main Theorem.

A Δ-$\mathscr{F}$-vector space V is said to split over an extension $\mathscr{F}_1$ of $\mathscr{F}$ if $V \otimes \mathscr{F}_1$ possesses an $\mathscr{F}_1$-basis (e_α) with $\delta e_\alpha = 0$ for all $\delta \in \Delta$. Start by recalling from [B, p. 79] a basic fact on splitting Δ-$\mathscr{F}$-vector spaces (cf. [T] for a generalisation to Hopf algebra actions more general than derivations).

(2.1) Lemma. *Any finite dimensional Δ-$\mathscr{F}$-vector space splits over some Picard–Vessiot extension of $\mathscr{F}$.*

For the sake of completeness we give the argument. If V is a Δ-$\mathscr{F}$-vector space with basis $e_1, ..., e_N$, write $\delta_k e_i = \sum a_{ij}^k e_j$.

Let $a^k = (a_{ij}^k)$ be viewed as an element of $\mathfrak{gl}_N(\mathscr{F})$. Commutativity of the δ_k's implies that $\delta_p a^k - \delta_k a^p + [a^k, a^p] = 0$ for all p and k. By Kolchin's surjectivity of the logarithmic derivative (cf. its form in [B1, p. 51, Corollary (2.9)]) there is a Picard–Vessiot extension $\mathscr{F}_1/\mathscr{F}$ and a matrix $g = (g_{ij}) \in GL_N(\mathscr{F}_1)$ such that $\delta_k g = a^k g$ for all k. Now the elements $f_1, ..., f_N$ of $V \otimes \mathscr{F}_1$ defined by $e_i = \sum g_{ij} f_j$ clearly form an $\mathscr{F}_1$-basis of the latter space and we easily check $\delta_k f_j = 0$ for all k and j.

The next lemma translates splittability in terms of local finiteness; recall that a Δ-$\mathscr{F}$-vector space V is called locally finite if it is a union of Δ-$\mathscr{F}$-vector spaces of finite dimension.

(2.2) Lemma. *Let G be a connected linear Δ-$\mathscr{F}$-group with* tr. deg. $\mathscr{F}\langle G\rangle/\mathscr{F} < \infty$. *Then the following are equivalent*:

(1) *G is splittable.*

(2) *G is splittable over some Picard–Vessiot extension of $\mathscr{F}$.*

(3) *The Δ-coordinate algebra $\mathscr{F}\{G\}$ is locally finite as a Δ-$\mathscr{F}$-vector space.*

Proof. $(2) \Rightarrow (1)$ is trivial.

$(1) \Rightarrow (3)$ Assume G is Δ-$\mathscr{F}_1$-isomorphic ($\mathscr{F}_1$ algebraically closed) with a split Δ-$\mathscr{F}$-group $H \subset GL_m(\mathscr{U})$. In order to prove that $\mathscr{F}\{G\}$ is locally finite as a Δ-$\mathscr{F}$-vector space it is sufficient to check that $\mathscr{F}_1\{G\}$ is locally finite as a Δ-$\mathscr{F}_1$-vector space. By (1.4) we have $\mathscr{F}_1\{G\} \simeq \mathscr{F}_1\{H\}$ so it is sufficient to check that $\mathscr{F}\{H\}$ is locally finite as a Δ-$\mathscr{F}$-vector space. Write $H = H^* \cap GL_m(\mathscr{K})$, hence $\mathscr{F}\{H\} = \mathscr{F}\{y\}_d/[\delta y, g_\alpha] = \mathscr{F}[y]_d/(g_\alpha)$, where $y = (y_{ij})$, $d = \det y$, and $g_\alpha \in \mathscr{C}[y]$; now conclude by noting that the $\mathscr{F}$-linear subspaces of $\mathscr{F}[y]/(g_\alpha)$ generated by monomials in y of bounded degree are Δ-$\mathscr{F}$-vector subspaces.

(3)$\Rightarrow$(2) There exists a finite dimensional Δ-$\mathscr{F}$-vector subspace V of $\mathscr{F}\{G\}$ generating $\mathscr{F}\{G\}$ as an $\mathscr{F}$-algebra. By Lemma (2.1), V splits over some Picard–Vessiot extension $\mathscr{F}_1$ of $\mathscr{F}$. It follows that the whole of $\mathscr{F}\{G\}$ splits over $\mathscr{F}_1$. So upon letting $\mathscr{R}=\mathscr{F}_1\{G\}$ we have $\mathscr{R}=\mathscr{R}^\Delta\otimes_{\mathscr{C}}\mathscr{F}_1$, where the upper Δ means "taking constants." Clearly $\mathscr{R}^\Delta$ is a finitely generated $\mathscr{C}$-subalgebra of $\mathscr{R}$. Since $(\mathscr{R}\otimes_{\mathscr{F}_1}\mathscr{R})^\Delta=\mathscr{R}^\Delta\otimes_{\mathscr{C}}\mathscr{R}^\Delta$, the comultiplication map $\mathscr{R}\to\mathscr{R}\otimes_{\mathscr{F}_1}\mathscr{R}$ takes $\mathscr{R}^\Delta$ into $\mathscr{R}^\Delta\otimes_{\mathscr{C}}\mathscr{R}^\Delta$ so $\mathscr{R}^\Delta$ becomes a finitely generated Hopf $\mathscr{C}$-algebra. Take any embedding $\mathscr{G}(\mathscr{R}^\Delta)\subset GL_n(\mathscr{C})$ and let H^* be the $\mathscr{C}$-closure of $\mathscr{G}(\mathscr{R}^\Delta)$ in $GL_n(\mathscr{U})$ and $H=H^*\cap GL_n(\mathscr{K})$. Then it is trivial to check that G is $\Delta-\mathscr{F}_1$-isomorphic with H. This closes the proof of the lemma.

Let us apply the implication (1)$\Rightarrow$(3) above to show that the Δ-subgroup G of $GL_1(\mathscr{U})$ defined by $y''y-(y')^2=0$ is not splittable. Indeed $\mathscr{F}\{G\}=\mathscr{F}[y, 1/y, y']$. Put $\gamma=y'/y$; then $\gamma'=0$ so for all $n\geqslant 0$, $y^{(n)}=\gamma^n y$, which shows that $\mathscr{F}\{G\}$ is not locally finite as a Δ-$\mathscr{F}$-vector space.

(2.3) Let V, W be Δ-$\mathscr{F}$-vector spaces. Recall that $V\otimes_{\mathscr{F}}W$ and $\mathrm{Hom}_{\mathscr{F}}(V, W)$ have natural structures of Δ-$\mathscr{F}$-vector spaces given by $\delta(x\otimes y)=(\delta x)\otimes y+x\otimes(\delta y)$ and $(\delta f)(x)=\delta(f(x))-f(\delta x)$ for $x\in V$, $y\in W$, $f\in\mathrm{Hom}_{\mathscr{F}}(V, W)$, $\delta\in\Delta$; in particular the linear dual $V^0=\mathrm{Hom}_{\mathscr{F}}(V,\mathscr{F})$ is a Δ-$\mathscr{F}$-vector space. Note that if V and W are locally finite, so is $V\otimes_{\mathscr{F}}W$; but $\mathrm{Hom}_{\mathscr{F}}(V, W)$ and V^0 need not be locally finite.

Now start with a finite dimensional Δ-Lie $\mathscr{F}$-algebra L. Then the universal enveloping algebra $U(L)$ inherits from the tensor algebra $\otimes(L)$ a structure of Δ-$\mathscr{F}$-algebra. So the dual $U(L)^0$ becomes a Δ-$\mathscr{F}$-vector space which is easily seen to be a Δ-$\mathscr{F}$-algebra with respect to convolution. Inside $U(L)^0$ lies the continuous dual $U(L)'$ (cf. [H, p. 228]); recall that $U(L)'$ is defined as the space of functionals whose kernel contains some two-sided ideal of finite codimension and that $U(L)'$ is a subalgebra of $U(L)^0$. One checks that $U(L)'$ is preserved by Δ: if $f\in U(L)'$ vanishes on an ideal J then δf must vanish on J^2. But even $U(L)'$ need not be locally finite (e.g., take L to be abelian of dimension $\geqslant 2$).

Next assume the radical L_r of L is nilpotent and denote it by R. Then in $U(L)'$ lies the algebra $\mathscr{B}(L)$ of R-nilpotent representative functions, which by definition is the space of all functionals in $U(L)'$ vanishing on some power of $R\cdot U(R)$ (cf. [H, p. 258]). We claim that $\mathscr{B}(L)$ is preserved by Δ. Indeed this follows from:

(2.4) Lemma. *If L is a Δ-Lie $\mathscr{F}$-algebra (of finite dimension), its radical R is a Δ-ideal.*

Proof. By (2.1), L splits over some Picard–Vessiot extension $\mathscr{F}_1$ so

$L\otimes\mathscr{F}_1=L_0\otimes_{\mathscr{C}}\mathscr{F}_1$, where $L_0=(L\otimes_{\mathscr{F}}\mathscr{F}_1)^{\Delta}$. Let R_0 be the radical of L_0. Then both $R_0\otimes_{\mathscr{C}}\mathscr{F}_1$ and $R\otimes_{\mathscr{F}}\mathscr{F}_1$ coincide with the radical of $L\otimes_{\mathscr{F}}\mathscr{F}_1$. Now $R=(R\otimes_{\mathscr{F}}\mathscr{F}_1)\cap L=(R_0\otimes_{\mathscr{C}}\mathscr{F}_1)\cap L$ and the latter space clearly is preserved by Δ.

(2.5) PROPOSITION. *If L is a Δ-Lie $\mathscr{F}$-algebra (of finite dimension) whose radical is nilpotent, $\mathscr{B}(L)$ is locally finite as a Δ-$\mathscr{F}$-vector space.*

Proof. First we claim that one can assume L splits over $\mathscr{F}$. Indeed by (2.1), L splits over some $\mathscr{F}_1$; suppose we know that $\mathscr{B}(L\otimes\mathscr{F}_1)=\bigcup V_\alpha$, where the V_α's are finite dimensional $\Delta-\mathscr{F}_1$-vector subspaces of $U(L\otimes\mathscr{F}_1)^0$. Then $\mathscr{B}(L)=\bigcup(V_\alpha\cap\mathscr{B}(L))$; but one checks that $\dim_{\mathscr{F}}(V_\alpha\cap\mathscr{B}(L))\leqslant\dim_{\mathscr{F}_1}V_\alpha$ and our claim is proved.

So assume $L=L_0\otimes_{\mathscr{C}}\mathscr{F}$, $L_0=L^{\Delta}$. Let $L_0=R_0+S_0$, where R_0 is the radical of L_0 and S_0 is a complementary semisimple Lie $\mathscr{C}$-algebra; then $R=R_0\otimes_{\mathscr{C}}\mathscr{F}$ is the radical of L and $S=S_0\otimes_{\mathscr{C}}\mathscr{F}$ is a complementary semisimple Lie $\mathscr{F}$-algebra, both R and S being Δ-$\mathscr{F}$-vector subspaces of L. Recall by [H, pp. 256–259] that the multiplication map $\mu:(U(L)')^R\otimes{}^S(U(L)')\to U(L)'$ is an isomorphism of $\mathscr{F}$-algebras, where $(U(L)')^R$ is the R-annihilated subalgebra of $U(L)'$ with respect to the left L-module structure of $U(L)'$ defined by $(x\cdot f)(u)=f(ux)$ $(x\in L$, $f\in U(L)'$, $u\in U(L))$ and ${}^S(U(L)')$ is the S-annihilated subalgebra of $U(L)'$ with respect to the right L-module structure of $U(L)'$ defined by $(f\cdot x)(u)=f(ux)$ $(x\in L$, $f\in U(L)'$, $u\in U(L))$. Moreover the following properties hold:

(1) The isomorphism μ induces an $\mathscr{F}$-algebra isomorphism $\tilde{\mu}:(U(L)')^R\otimes{}^S(\mathscr{B}(L))\to\mathscr{B}(L)$,

(2) $(U(L)')^R$ coincides with the image of the natural injection $\alpha: U(S)'\to U(L)'$, and

(3) The restriction map $U(L)'\to U(R)'$ induces an isomorphism $\beta:{}^S(\mathscr{B}(L))\to\mathscr{B}(R)$, where $\mathscr{B}(R)$ is the algebra of R-nilpotent representative functions on $U(R)$.

Since S is a Δ-subalgebra of L it follows that ${}^S(U(L)')$ is a Δ-$\mathscr{F}$-vector subspace of $U(L)'$ so one sees that the map μ is a Δ-map, hence so is $\tilde{\mu}$. Since α and β above are obviously Δ-maps it follows that the induced isomorphism of $\mathscr{F}$-algebras $\mathscr{B}(L)\simeq U(S)'\otimes\mathscr{B}(R)$ is a Δ-map. So it is sufficient to check that each of $U(S)'$ and $\mathscr{B}(R)$ is locally finite as a Δ-$\mathscr{F}$-vector space. Now $\mathscr{B}(R)=\bigcup V_n$, where V_n is the subspace of all functionals on $U(R)$ vanishing on $(R\cdot U(R))^n$; clearly V_n are finite dimensional Δ-$\mathscr{F}$-vector subspaces of $\mathscr{B}(R)$. To check the assertion for $U(S)'$ we prove the following (a priori) more general:

(2.6) LEMMA. *Let S be a Δ-Lie $\mathcal{F}$-algebra (of finite dimension). Assume that for any S-module V of finite dimension we have* $\mathrm{Ext}^1_S(V, V)=0$. *Then $U(S)'$ is locally finite as a Δ-$\mathcal{F}$-vector space.*

Proof. We have $U(S)' = \bigcup V_J$, where J runs through the set Σ of all two-sided ideals of finite codimension and $V_J = \{f \in U(S)'; f(J)=0\}$. We shall be done if we show that the V_J's are preserved by Δ. For this it is sufficient to check that any ideal $J \in \Sigma$ is a Δ-ideal.

Let $J \in \Sigma$, put $N = \dim_{\mathcal{F}} U(S)/J$, let $V = \mathcal{F}^N$ viewed with its natural structure of Δ-$\mathcal{F}$-vector space, and fix an $\mathcal{F}$-linear isomorphism $V \simeq U(S)/J$. Moreover consider the algebra map $\varphi : U(S) \to \mathrm{End}(V)$ which takes any $u \in U(S)$ into the endomorphism of V corresponding to the multiplication from the left by u in $U(S)/J$; clearly $\ker \varphi = J$. Now φ restricted to S yields a representation $\rho : S \to \mathfrak{gl}(V)$. Since $\mathrm{Hom}_{\mathcal{F}}(S, \mathfrak{gl}(V))$ is a Δ-$\mathcal{F}$-vector space we may consider for any $\delta \in \Delta$ the linear map $\delta\rho \in \mathrm{Hom}_{\mathcal{F}}(S, \mathfrak{gl}(V))$. It is easy to check that $\delta\rho$ are in fact cocycles for S in $\mathfrak{gl}(V)$, where $\mathfrak{gl}(V)$ is viewed as an S-module via the representation $S \to^{\rho} \mathfrak{gl}(V) \to^{\mathrm{ad}} \mathfrak{gl}(\mathfrak{gl}(V))$. Since $\mathrm{Ext}^1_S(V, V) = H^1(S, \mathfrak{gl}(V))$ is assumed to vanish, $\delta\rho$ must be coboundaries so there exist $h_1, ..., h_m \in \mathfrak{gl}(V)$ such that for any $x \in S$,

$$\delta_i(\rho(x)) - \rho(\delta_i x) = [\rho(x), h_i].$$

For each index i consider the $\mathcal{F}$-linear maps $D_1, D_2 : U(S) \to \mathrm{End}(V)$ defined by $D_1(u) = \delta_i(\varphi(u)) - \varphi(\delta_i u)$ and $D_2(u) = \varphi(u)h_i - h_i\varphi(u)$. One checks that both D_1 and D_2 are φ-derivations, i.e., satisfy the formula

$$D(uv) = D(u)\,\varphi(v) + \varphi(u)\,D(v), \qquad u, v \in U(S),$$

where $D = D_1, D_2$. Since D_1 and D_2 agree on S they agree on all of $U(S)$; but this shows that if $\varphi(u)=0$ for some $u \in U(S)$ then $\varphi(\delta_i u) = 0$. Since this holds for all indices i, $\ker \varphi$ is a Δ-ideal and we are done.

(2.7) Next we relate groups and Lie algebras. Start with an irreducible linear Δ-$\mathcal{F}$-group G and let $\mathcal{G} = \mathcal{G}(\mathcal{F}\{G\})$, so $\mathcal{P}(\mathcal{G}) = \mathcal{F}\{G\}$. Let us put a structure of Δ-Lie $\mathcal{F}$-algebra on $\mathcal{L}(\mathcal{G})$ as follows. First consider the Δ-$\mathcal{F}$-vector space structure on $\mathcal{P}(\mathcal{G})^0$; next check that with respect to convolution $\mathcal{P}(\mathcal{G})^0$ becomes a Δ-$\mathcal{F}$-algebra, hence $\mathcal{L}(\mathcal{P}(\mathcal{G})^0)$ becomes a Δ-Lie $\mathcal{F}$-algebra. Finally, check that $\mathcal{L}(\mathcal{G})$ (which is defined as a Lie subalgebra of $\mathcal{L}(\mathcal{P}(\mathcal{G})^0)$; cf. [H, p. 36]) is preserved by Δ. From this construction we see that the naturally induced embedding $e_{\mathcal{G}} : \mathcal{P}(\mathcal{G}) \to U(\mathcal{L}(\mathcal{G}))'$ [H, p. 230] is a Δ-algebra map.

(2.8) LEMMA. *Let $\mathcal{G}_r$ be the radical of $\mathcal{G}$ and G_r be the radical of G. Then*:

(1) *The defining ideal of* $\mathscr{G}_r$ *in* $\mathscr{P}(\mathscr{G})$ *is a* Δ*-ideal.*

(2) $\mathscr{G}(\mathscr{U}\{G_r\}) = \mathscr{G}(\mathscr{U}\{G\})_r$.

(3) G_r *is unipotent if and only if* $\mathscr{G}_r$ *is unipotent.*

Proof. (1) Consider the embeddings $e_{\mathscr{G}}$ and $e_{\mathscr{G}_r}: \mathscr{P}(\mathscr{G}_r) \to U(\mathscr{L}(\mathscr{G}_r))'$ as inclusions. Then the defining ideal of $\mathscr{G}_r$ in $\mathscr{P}(\mathscr{G})$ is precisely the intersection (taken in $U(\mathscr{L}(\mathscr{G}))'$) of $\mathscr{P}(\mathscr{G})$ with the kernel of the map

$$\pi: U(\mathscr{L}(\mathscr{G}))' \to U(\mathscr{L}(\mathscr{G}_r))' = U(\mathscr{L}(\mathscr{G})_r)'.$$

But since by (2.4), $\mathscr{L}(\mathscr{G})_r$ is a Δ-ideal in $\mathscr{L}(\mathscr{G})$, π is a Δ-map and we are done.

(2) We have group inclusions

$$\begin{array}{ccc} G_r & \subset & \mathscr{G}(\mathscr{U}\{G_r\}) \\ \cap & & \cap \\ G & \subset & \mathscr{G}(\mathscr{U}\{G\}) \end{array}$$

From the fact that G_r (respectively G) is Zariski dense in $\mathscr{G}(\mathscr{U}\{G_r\})$ (respectively in $\mathscr{G}(\mathscr{U}\{G\})$), it follows immediately that $\mathscr{G}(\mathscr{U}\{G_r\})$ is an irreducible normal solvable subgroup of $\mathscr{G}(\mathscr{U}\{G\})$ hence it is contained in $\mathscr{G}(\mathscr{U}\{G\})_r$. On the other hand, by assertion (1) the Δ-$\mathscr{F}$-group $G' = \mathscr{G}(\mathscr{U}\{G\})_r \cap G$ is irreducible and dense in $\mathscr{G}(\mathscr{U}\{G\})_r$. Clearly G' is normal in G and solvable so $G' \subset G_r$. Taking Zariski closure we get $\mathscr{G}(\mathscr{U}\{G\})_r \subset \mathscr{G}(\mathscr{U}\{G_r\})$ and we are done.

(3) If G_r is unipotent, $\mathscr{U}\{G\}$ is locally unipotent as a G_r-module [H, p. 65] so it will also be so as a $\mathscr{G}(\mathscr{U}\{G\})_r$-module by assertion (2). So $\mathscr{G}(\mathscr{U}\{G\})_r$ (and hence also $\mathscr{G}_r$) is unipotent. The converse is obvious.

(2.9) We are in a position to conclude the proof of the Main Theorem. Indeed if G_r is unipotent, by Lemma (2.8) above $\mathscr{G}$ has a unipotent radical. By [H, p. 260] the image of $\mathscr{P}(\mathscr{G})$ via the map $e: \mathscr{P}(\mathscr{G}) \to U(\mathscr{L}(\mathscr{G}))'$ is contained in $\mathscr{B}(\mathscr{L}(\mathscr{G}))$. Since by Proposition (2.5), $\mathscr{B}(\mathscr{L}(\mathscr{G}))$ is locally finite as a Δ-$\mathscr{F}$-vector space, so will be $\mathscr{P}(\mathscr{G})$ and we may conclude by Lemma (2.2).

3. Final Remark

In proving our Lemma (1.2) we in fact proved the following useful "dévisage" property: let $\mathscr{A} \subset \mathscr{B}$ be an extension of integral Δ-$\mathbb{Q}$-algebras such that $\mathscr{B}$ is Δ-generated over $\mathscr{A}$ by one element; then there exist a non-zero element $s \in \mathscr{B}$ and a (non-differential) sub-$\mathscr{A}$-algebra R of $\mathscr{B}[1/s]$

such that R is a polynomial $\mathscr{A}$-algebra (in possibly infinitely many variables) and $\mathscr{B}[1/s]$ is finitely generated as a (non-differential) R-algebra. Here is an application. Let $t \in \mathscr{B}$, $t \neq 0$; since $\mathscr{B}[1/st]$ is finitely generated as an R-algebra there is a non-zero element $F \in R$ such that any prime in R not containing F is the trace on R of some prime in $\mathscr{B}[1/t]$. Viewing F as a polynomial with coefficients in $\mathscr{A}$ and picking any non-zero coefficient f of it we get that any prime P in $\mathscr{A}$ not containing f is the trace on $\mathscr{A}$ of some prime Q in $\mathscr{B}$ not containing t, i.e., the ring $B[1/t] \otimes_{\mathscr{A}} Q(\mathscr{A}/P)$ is non-zero. But if P is a Δ-ideal the latter ring is a Δ-$\mathbb{Q}$-algebra hence possesses at least one prime Δ-ideal. Consequently Q above can be chosen to be a Δ-ideal. Using an obvious induction we get a quite elementary and short proof of Seidenberg's theorem on "extending differential specialisations" (cf. [K1, p. 140] for an arbitrary characteristic generalisation) saying if $\mathscr{A} \subset \mathscr{B}$ is a Δ-finitely generated extension of integral Δ-$\mathbb{Q}$-algebras then for any non-zero $t \in \mathscr{B}$ there exists a non-zero $f \in \mathscr{A}$ such that any prime Δ-ideal in $\mathscr{A}$ not containing f is the trace in $\mathscr{A}$ of some prime Δ-ideal in $\mathscr{B}$ not containing t. Now exactly as in [B2] this implies a "differential Chevalley constructibility theorem."

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